Transmission-Line Properties of Parallel Strips Separated by a Dielectric Sheet

HAROLD A. WHEELER, FELLOW, IEEE

Abstract—A transmission line is made of a symmetrical pair of strip conductors, or a single strip and a ground plane, on opposite faces of a sheet of dielectric material. There is computed, to a close approximation, the relations among the dielectric constant of the sheet, the effective dielectric constant of the sheet and the empty space, the shape ratio, and the wave resistance, for the entire range of possible values. These relations are summarized in a graphical chart covering the range of practical interest.

The computation is based on conformal mapping of the dielectric boundary on coordinates such that its effect can be most closely evaluated by simple principles. All relations are approximated in terms of ordinary functions (exponential and hyperbolic). Of particular interest is the effective filling fraction of the dielectric material, which depends mainly on the shape ratio and only slightly on the dielectric constant. Explicit formulas are given for analysis or synthesis.

I. Introduction

IN THE DECADE of 1945 to 1955, there was much interest in the subject of electromagnetic-wave transmission lines formed of strips of sheet conductor in various configurations. This culminated in a special issue of the IRE Transactions on Microwave Theory and Techniques in March 1955.

In a previous paper by the author [1], it was pointed out that one simple configuration had been resistant to solution in useful form. This is the symmetrical case of parallel flat strips, face-to-face, or the corresponding asymmetrical case of one flat strip parallel to a ground plane. That paper has presented a solution of this case, in useful form, for strip width comparable with the separation, or greater. That solution is limited to a wave medium of uniform properties, such as free space.

There remains the need for a useful solution for such parallel strips, separated by a sheet of dielectric material serving to support the strips (or one strip relative to a ground plane). This problem of mixed dielectric (air and sheet) has received little attention in the literature, probably because it is difficult to solve by the usual methods. The writer has developed an approach to this problem, which is found to yield close approximations in terms of simple mathematical functions (slide-rule functions, no more advanced than exponential and hyperbolic).

It is the purpose of this paper to present a solution to this problem, leading to graphical charts covering a wide range of shape ratio of the strips and of dielectric constant of the sheet material. The solution employs conformal mapping, based largely on the preceding article [1]. It serves as an introduction to a particular concept for the solution of any problem of mixed dielectric, using the present problem as an example.

It is required to express a relation between the transmission-line properties, such as wave resistance and the shape ratio (strip-width/separation), this relation depending on the dielectric constant of the sheet material. The solution will be developed separately for two ranges of shape ratio, characterized by strips "wide" or "narrow" relative to their separation.

The present approach is based on several steps involving conformal mapping. First, assuming the strips to be surrounded by only one kind of dielectric (free space), the actual field configuration is mapped on a rectangular area of uniform field. Secondly, the straight boundary of the dielectric sheet is mapped as a curve on this rectangle, so the presence of this dielectric distorts the field. Thirdly, some rules are perceived for approximating the effective filling fraction of the dielectric material, from which the effective dielectric constant can be simply computed.

This solution in its complete form does not yield an explicit formula for either synthesis or analysis, but is well adapted for computing a graphical chart for interpolation. However, it is found possible to devise explicit formulas for synthesis or analysis, based on the extremes of "wide" and "narrow" strips, which are useful approximations.

The conformal mapping will be introduced with reference to the two planes here involved, the plane of space coordinates and the plane of flux-potential coordinates (based on free space). Then there will be presented the concepts of approximation for mixed dielectric. The solution of the problem will be developed separately for "wide" strips and for "narrow" strips. The former will be based on the preceding monograph [1]. The latter will rely on some unpublished studies by the author. After some general discussion of significant relations, some procedures for computation, will be given followed by graphical charts in a form convenient for practical applications.

The derivation is based essentially on the separate evaluation of inductance and capacitance along the line. This is valid if substantially all of the energy is in the TEM mode of propagation, the operating frequency being far below cutoff in all higher modes. This condi-

Manuscript received August 11, 1964; revised November 13, 1964.
The author is with Wheeler Laboratories, Great Neck, N. Y.

tion is met if the transverse dimensions are much less than $\frac{1}{2}$ wavelength in the dielectric material.

II. Conformal Mapping of Dielectric Boundary

Figure 1 shows the cross section of two parallel strips on opposite faces of a dielectric sheet. The essential dimensions are the strip width $2a$, the separation equal to the thickness of the dielectric sheet $2b$, and the dielectric constant of the sheet k. The strips form a balanced transmission line having certain values of wave resistance R and effective dielectric constant k'. The strips are assumed to be made of thin sheets of perfect conductor. The dielectric material is assumed to be homogeneous, isotropic and free of dissipation, so it is completely described by one constant k.

The same parallel strips, but without the dielectric sheet, can be formulated more simply, since the space is filled with only one kind of dielectric. For the case of "wide" strips, this is the subject of the writer's preceding monograph [1], which provides much background for the present treatment; equations therein are referenced by their number with this prefix (A- · · ·).

One quadrant of the cross section of the line can be represented on one quadrant of space coordinates (the plane of $x+jy=z$). This is shown in Fig. 2, which is similar to the corresponding Fig. 2 in [1]. Here we add the intervening sheet of dielectric k. Attention is directed to the free-space flux line between points ③ and ⑦. This is the line which would terminate on the edge of the strip if the dielectric sheet were not present. The dielectric outside of this line, in the shaded area, causes a distortion of the flux lines, which complicates the effect of the dielectric sheet. The principal objective in this article is the evaluation of this effect.

By the procedure described in [1], the conductor boundaries are mapped on the z' plane of flux and potential coordinates, as shown in Fig. 3(a). The same symbols are retained. Furthermore, the boundary of the dielectric sheet is mapped on this plane as a curve between points ③ and ⑧. This operation reduces the problem to one of mixed dielectric in the space between parallel-plane conductors of unlimited extent. The area mapped in Fig. 3(a) is one quadrant of the cross section as shown in Fig. 2. The related diagram in Fig. 3(b) will be explained further on.

In Fig. 3(a), the area inside of the curve represents the region outside of the dielectric sheet, bounded by points ①, ②, ③, and ⑧. The area outside of the curve represents the region occupied by the dielectric sheet, bounded by points ③, ④, ⑥, and ⑧. It is required to evaluate the effective dielectric constant of the entire region, which will be a weighted mean between the dielectric constant of the material and unity for free space.

It is noted that the conformal mapping of a boundary between two dielectrics is valid because it retains the angles of "refraction" of the electric field at the boundary.

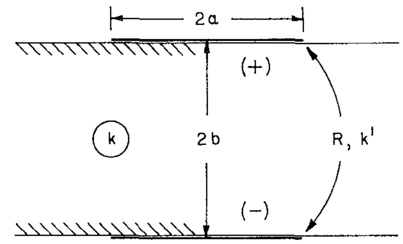


Fig. 1. Transmission line formed of parallel strips on opposite faces of dielectric sheet.

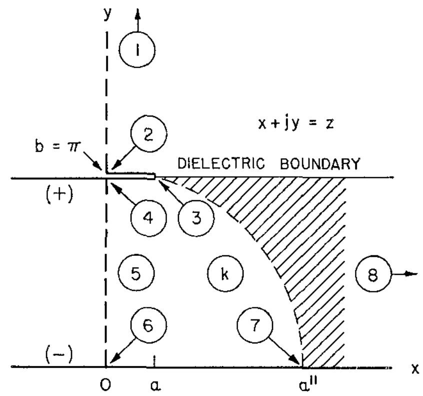


Fig. 2. Space coordinates showing dielectric sheet.

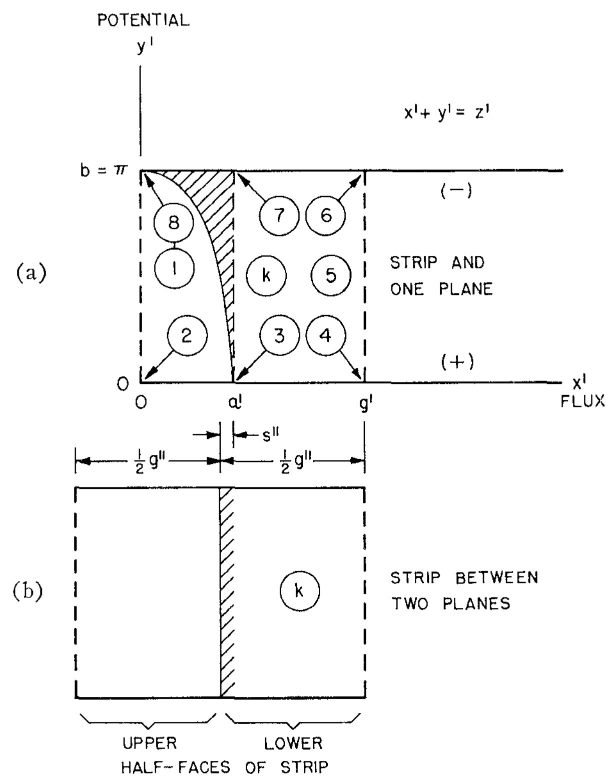


Fig. 3. Flux-potential coordinates, showing dielectric boundary.

III. Principles of Approximation for Mixed Dielectrics

Two parameters are required to determine any example in the present study; these may be taken as the shape ratio a/b and the sheet dielectric constant k. In terms of these parameters, all examples may be charted on a square region as in the diagram of Fig. 4. Every point in this region has a value of effective dielectric constant k' between the limits indicated. For the center of this diagram, it is found that $k'=1.70$, which is between the limits 1.5 and 2.

On this square diagram, definite coordinate scales could be assigned and then contours of k' could be plotted. In Fig. 4, the scales are so chosen that the contour through the center is nearly a straight line. Since the top and bottom contours are horizontal straight lines, it is conjectured that all the intermediate contours would be nearly straight lines of intermediate slopes.

The principal purpose of Fig. 4 is to indicate the limits of the principal parameters and the ranges of particular interest. To this end, the shape ratio is designated as follows for various ranges, in terms of strip width relative to separation.

Very narrow:	$a/b \ll 1$
Narrow:	$a/b < 1$
Square:	$a/b = 1$
Wide:	$a/b > 1$
Very wide:	$a/b \gg 1$

Then there are simple rules for the extreme values of effective dielectric constant k' corresponding to the extremes of shape.

Very wide: $k' = k$

Very narrow: $k' = k_a = \dfrac{k+1}{2}; \quad k = 2k' - 1$

The former is based on the principle that nearly all of the flux is in the dielectric. The latter is based on the principle that the electric field is symmetrically concentrated near the strip, with free space on one side and dielectric on the other side.

Another parameter is introduced, for describing the partial filling of dielectric. This is the effective filling fraction q which is to be evaluated in this presentation. This fraction is mainly dependent on the shape. For any particular shape, its value lies between an upper bound q' and a lower bound q'', respectively, corresponding to the extremes of lo-k and hi-k. These bounds are so close that one suffices for fairly close approximation, preferably the latter q'' because the resulting effect of the dielectric is greater for hi-k.

Referring still to Fig. 4, the present problem may be regarded as one of determining the extreme conditions on the square diagram, and then interpolating for intermediate conditions, in order to evaluate the effective dielectric constant for all conditions. This is to be accomplished for formulating the bounds q', q'' of the effective filling fraction q and then interpolating for intermediate values of dielectric constant k.

There are only two cases of mixed dielectric that can be directly evaluated by conformal mapping. In one case, the dielectric boundaries are all located on flux lines; in the other case, on potential contours. On the coordinates of flux and potential, as in Fig. 3(a), every such boundary would be parallel to one axis or the other. The curve representing the actual boundary meets neither of these conditions, but it can be interpreted as a mixture of both conditions. To develop this idea, the same region is expanded in Fig. 5(a), showing the curve representing the dielectric boundary.

First we note that a certain rectangle of space $a'<x'<g'$ is filled with dielectric. It happens that this is always more than half of the space as mapped on the z' plane. Then we note that the remaining rectangle $0<x'<a'$ is mostly empty of dielectric, the exception being the shaded area in Fig. 3(a) or 5(a). It is this extra area of dielectric which defies exact analysis but will be found susceptible of close approximation in its effect.

The shaded area, just outside of the curved boundary in Fig. 5(a) can be divided into two parts, as shown in Fig. 5(b). The entire area is expressed as $\pi s'$, in which s' is its effective width on the x' axis. One part of the area is expressed as $\pi s''$, which effectively adds s'' to the width of the dielectric region on the right; this is termed the "parallel" component. The remainder of the area is $\pi(s'-s'')$, which is effectively in "series" with the free-space region outside of the dielectric. This analysis gives a close approximation of the effect of the shaded area of dielectric, which is a small part of the total effect of the dielectric.

From the viewpoint of Fig. 5, the effective width s of the shaded area is between two bounds s', s''. A simple rule for interpolation is to be formulated here. In Fig. 5(b), the effect of the areas can be formulated in terms of dielectric and air capacitors in series and parallel connections. As a first approximation for small shaded area, the resultant effective width is found to be

$$s = s'' + \frac{s'-s''}{k} = \frac{1}{k}s' + \frac{k-1}{k}s'' \qquad (1)$$

In words, the parallel component of shaded area is fully effective while the series component is $1/k$ as effective. While this rule is based on simplifying assumptions, it is probably valid as a close approximation for the curved boundary.

In Figs. 3(a) and 5(a), the filling fraction is defined as the ratio of dielectric area over total area in the rectangle of field mapping. It is most easily formulated in

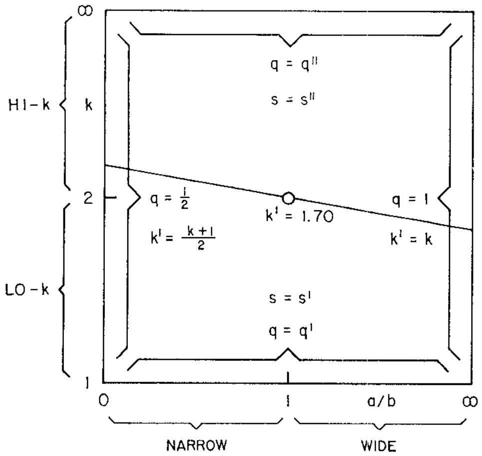


Fig. 4. Diagram of entire range of shape ratio and dielectric constant.

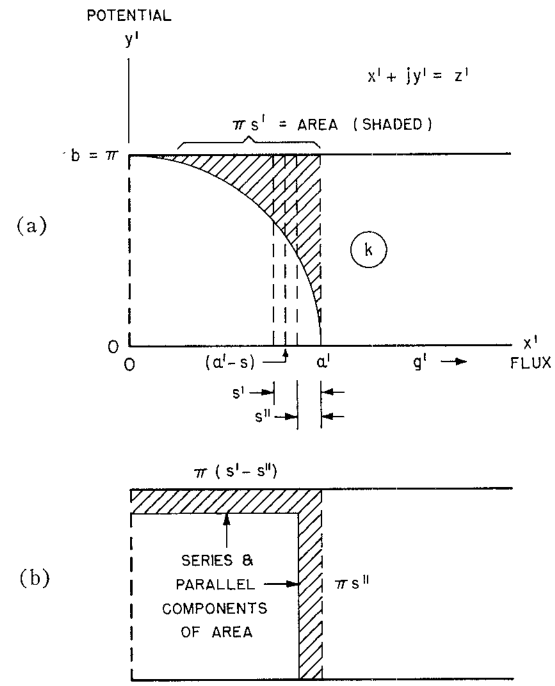


Fig. 5. Extra effective width analyzed in terms of parallel and series components.

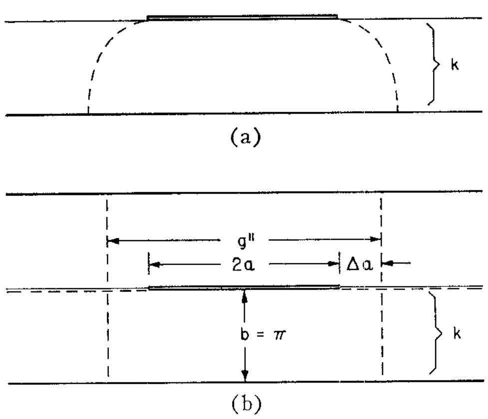


Fig. 6. Comparison with strip between parallel planes.

terms of effective width. The actual filling fraction is

$$q' = \frac{g' - a' + s'}{g} = 1 - \frac{a' - s'}{g'} = 1 - \frac{a'}{g'}\left(1 - \frac{s'}{a'}\right) \quad (2)$$

This includes all of the area outside of the curve, which is the area filled with dielectric. The "parallel" part of the filling fraction is only slightly less.

$$q'' = \frac{g' - a' + s''}{g'} = 1 - \frac{a' - s''}{g'} = 1 - \frac{a'}{g'}\left(1 - \frac{s''}{a'}\right)$$

$$= q' - \frac{s' - s''}{g'} \quad (3)$$

The effective filling fraction is found by (1).

$$q = \frac{g' - a' + s}{g'} = 1 - \frac{a' - s}{g'} = q'' + \frac{q' - q''}{k}$$

$$= q'' + \frac{s' - s''}{kg'} \quad (4)$$

The required components (s', s'' or q', q'') remain to be evaluated by various methods to be described further on.

The relation between the effective filling fraction q and the effective dielectric constant k' can be stated from the concept of parallel capacitors.

$$q = \frac{k' - 1}{k - 1} \quad (5)$$

$$k' = (1 - q) + qk = 1 + q(k - 1)$$

$$= k - (k + 1)\frac{a'}{g'}\left(1 - \frac{s''}{a'} - \frac{s' - s''}{ka'}\right) \quad (6)$$

The last form is in terms of the free-space flux ratio a'/g' from [1].

Referring to Figs. 3(a) and 5(a), there are simple concepts that are valid for the extremes of dielectric constant k. In the "lo-k" extreme, the dielectric does not cause appreciable distortion of the field. Therefore the dielectric area is entirely effective, but its effect is small, being proportional to $k-1$. In the "hi-k" extreme, there is no appreciable energy outside of the dielectric area. Therefore only the "parallel" part of the area is effective. The entire area is defined by conformal mapping of the dielectric boundary, and can be computed to some degree of approximation. The parallel part can be defined and computed by another case of conformal mapping to be described here.

Figure 6(a) shows the difference between the dielectric-sheet boundary and the free-space flux line from the edge of the strip. Figure 6(b) shows another configuration in which the boundary and the flux line coincide [3]. This is accomplished by adding above the strip an image of the neutral plane below the strip. In the resulting configuration of a strip between parallel planes, the effect of the dielectric sheet can be evaluated simply and exactly. In the limiting case of hi-k, these two cases become identical, because all of the flux is in the dielectric. In one quadrant of Fig. 6(a), as represented in Fig. 3(a), the effective width of the dielectric filling is between $\frac{1}{2}g'$ and g'. In the corresponding quadrant of Fig. 6(b), as represented in Fig. 3(b), the entire effective width is just half filled with dielectric, so the effective width of the dielectric filling is $\frac{1}{2}g''$. The latter can be exactly described and evaluated, thereby providing an evaluation of the former. Formulas for the latter are given in the last section herein for reference. The lower bound of the effective filling fraction becomes

$$q'' = \tfrac{1}{2}g''/g' > \tfrac{1}{2} \tag{7}$$

As previously stated, the principal problem in this derivation is the evaluation of the shaded area outside of the curved contour in Figs. 3(a) and 5(a). This will be computed for various cases, as a fraction of the rectangle in which the curve is inscribed. This "area fraction," s'/a', is graphed in Fig. 7, showing its variation with shape ratio. The curve is found to be always outside of an inscribed elliptic quadrant, so the area fraction is less than that for an ellipse, $1-\pi/4$, as seen from the graph of s' in Fig. 7. (This will be discussed further with reference to Fig. 8.) The area fraction has a maximum value near the "square" shape, and decreases toward zero for both extreme shapes. It is plotted on a special scale of abscissas, so chosen that this graph has a simple shape close to a sine wave. This scale and the properties of the graph will be explained in the course of the derivations and computations. The lower graph s'' is somewhat similar in shape, and represents the parallel component of the area fraction s''/a'. The effective area fraction s/a' is a weighted mean between these two graphs, the weighting depending on the dielectric constant k as indicated.

In developing mathematical approximations for the entire range of the parameters, it is helpful to work from the extremes of shape to some more elaborate formulas for intermediate shapes. Therefore this approach will be exploited in the derivations to follow. For wide strips, the derivation is a continuation of [1]. For narrow strips, the required background will be given without proof.

In the literature, the closest reference found by the writer is the 1956 and 1958 papers by Dukes [3], [4]. He does approach the problem of mixed dielectric in the configuration of a strip and ground plane separated by a dielectric sheet. His principal contribution is the concept of Fig. 6(b) herein, which gives the asymptotic solution for high dielectric constant. This concept should have led to fairly close approximations, which he expected. However, he seems to have made an error in application of this principle, since his main formula contains a fallacy in relations. His graphs, published in the later paper, contain errors as great as 15 per cent of the wave resistance. The form of his graphs is followed here for wave resistance in Fig. 9.

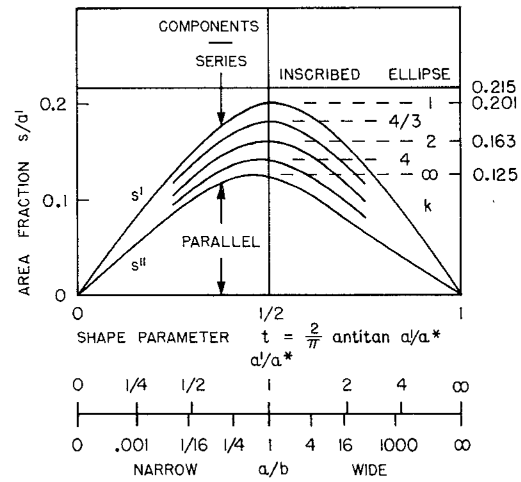


Fig. 7. Variation of extra area with shape.

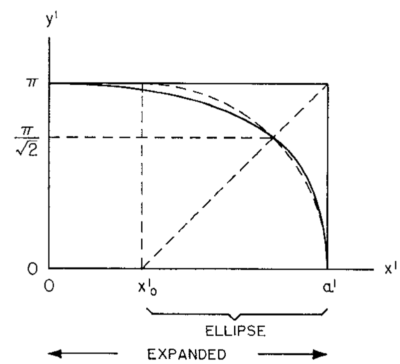


Fig. 8. Approximation of extra area by ellipse.

IV. Derivation for Wide Strips

This derivation for wide strips is based on the preceding monograph [1], which will be duplicated here only to the extent desired for continuity. As mentioned before, equations in that paper will be identified by the prefix A followed by the number (A- · · ·).

As in [1], the effective half-width g' on the x' plane is related to the free-space wave resistance R_1, the inductance L and the free-space capacitance C_1.

$$\begin{aligned} R_1 &= R_c\pi/g'; & g' &= \pi R_c/R_1 \\ L &= \mu_0 l\pi/g'; & g' &= \pi\mu_0 l/L \\ C_1 &= \epsilon_0 l g'/\pi; & g' &= \pi C_1/\epsilon_0 l \end{aligned} \tag{8}$$

(The symbols are listed in a later section.) In the present problem of mixed dielectric, a significant parameter is the effective dielectric constant k' of the entire space which is partially filled with the dielectric material k. The resulting values of wave resistance and capacitance are

$$R = \frac{1}{\sqrt{k'}} R_1; \qquad C = k'C_1 \tag{9}$$

Also the wave velocity is decreased in the same ratio as the resistance.

The marginal condition for "wide" strips (the "square" condition, $a/b=1$) corresponds roughly to $R=\frac{1}{2}R_c/\sqrt{k'}$, for which $g'=2\pi$. Therefore this value of g' may be taken as representative of this condition. It happens that the "wide" approximations are useful for even narrower strips, so the case of $g'=\pi$ may be taken as the extreme condition for the "wide" formulas.

The conformal-mapping formulas are given in [1], for the relation between the space coordinates on the z plane of Fig. 2 and the flux-potential coordinates on the z' plane of Fig. 3(a). Some of the formulas to be presented here are more simply expressed directly in terms of the parameter d, which is nearly equal to the effective width g' and therefore may be used interchangeably in all except the closest approximations. Complete computations, especially in the marginal region, may include the small difference between these two parameters $d-g'$.

The mapping approximation in [1] is simple enough to enable explicit formulation of the curve in Figs. 3(a) and 5(a), representing the dielectric boundary on the flux-potential coordinates. Here several parameters d, d', g' are taken as equal. From (A-52) and (A-53),

$$z = x + jy = j\pi + d \tanh \tfrac{1}{2}z' - z' \qquad (10)$$

At the dielectric boundary,

$$z = j\pi; \qquad x = 0; \qquad y = \pi \qquad (11)$$

Separating the real and imaginary parts of (10), the imaginary part determines a curve on the z' plane, representing the dielectric boundary.

$$y = \pi = \pi + d\frac{\tan \frac{1}{2}y'(1 - \tanh^2 \frac{1}{2}x')}{1 + \tan^2 \frac{1}{2}y' \tanh^2 \frac{1}{2}x'} - y' \qquad (12)$$

Here it is possible to express x' of y' explicitly in closed form. By routine transformations, this is reduced to the following simple form:

$$\cosh x' = d\frac{\sin y'}{y'} - \cos y' \qquad (13)$$

The shaded area is found to approach an upper limit for the "very wide" condition of increasing d and a', where the mapping approximation approaches perfection. This is based on the following integration carried to the limit of increasing d; the integrand is based on (13) for the entire curve.

$$a' - x' = \ln \frac{\dfrac{\sin y'}{y'} - \dfrac{\cos y'}{d}}{1 - 1/d} = \ln \frac{\sin y'}{y'} \qquad (14)$$

$$\max s' = \frac{1}{\pi}\int_0^\pi (a' - x')dy' = 2 \ln \frac{\pi}{2} = 0.903 \qquad (15)$$

In the marginal "wide" condition, near the "square" shape, it is found that the dielectric boundary curve (13) comes closest to the shape of an elliptic quadrant inscribed in the same rectangle. This fact is utilized to obtain a close approximation for the area fraction, as will be developed with reference to Fig. 8. Outside of the actual curve, or of an ellipse, most of the area is near the corner of the rectangle. Therefore an ellipse is specified which crosses the actual curve near the corner. This ellipse is one inscribed in a rectangle which is smaller than the entire rectangle, having a lesser dimension along the x' axis. This dimension is chosen so that the ellipse crosses the actual curve at the point on the diagonal of the ellipse, as shown. Then the area outside of this ellipse is taken as an approximation for the area outside of the actual curve.

On this basis, the point of intersection is located as indicated in Fig. 8. The area outside of the curve is taken to be that outside of the ellipse. The resulting effective width can be expressed in terms of a', as follows:

$$s' = 0.732[a' - \text{anticosh}\,(0.358 \cosh a' + 0.953)] \qquad (16)$$

This formula for the effective width might not give a close approximation for the "very wide" case, so it may be compared with (15) for this case.

$$(16): \quad \max s' = 0.732 \ln \frac{1}{0.358} = 0.752 \qquad (17)$$

$$(15): \quad \max s' = 2 \ln \frac{\pi}{2} = 0.903 \qquad (18)$$

The fact that the ellipse approximation comes so close ($\frac{5}{6}$ of the correct value) in the "very wide" limit is an indication of its probable very close approximation in the intermediate "wide" range for which it is intended (including the "square" shape).

The area fraction is found to have a maximum value near the "square" condition, which has the following properties (see [1], Table I). If

$$a/b = 1: \qquad g' = d = 6.65; \qquad a' = 2.416 \qquad (19)$$

A slightly wider condition gives the maximum area fraction, which may be compared with that outside of an inscribed elliptic quadrant.

$$(16): \quad \max s'/a' = 0.201$$
$$\text{Ellipse: } (1 - \pi/4) = 0.215 \qquad (20)$$

The area fraction is seen to be always somewhat less than that outside of an elliptic quadrant inscribed in the entire rectangle. (This is not the same as the elliptic quadrant used to approximate the curve, which is inscribed in a smaller rectangle.)

Referring to Fig. 7, formula (16) approximates the area fraction in the intermediate range, while (15) evaluates the limiting slope in the "very wide" range. The full development of this graph is still deferred for more background.

With reference to Figs. 5 and 6, there has been introduced the concept of series and parallel components of the shaded area just outside of the dielectric-boundary curve on the z' plane. The parallel component has an effective area $\pi s''$ and effective width s'' somewhat less than the corresponding values $\pi s'$ and s' for the entire shaded area. The parallel component is to be formulated from the configuration in Fig. 6(b).

As previously explained, the lower half of Fig. 6(b) is a representation of the space occupied by the dielectric sheet k. This set of boundaries can be evaluated exactly, as described in the last section herein. Here, it seems preferable to use the simple close approximation for the "wide" case. The fringing field at each edge causes the effective half-width $\frac{1}{2}g''$ to exceed the actual half-width a by a small amount Δa. This excess approaches a limit very rapidly as the strip width approaches and exceeds the spacing, the limiting value for "very wide" strips being

$$\Delta a = \tfrac{1}{2}g'' - a = \ln 4 = \frac{b}{\pi} \ln 4 \tag{21}$$

Referring to Fig. 3, the dielectric space mapped from Fig. 2 is the same as that in Fig. 6(b). The region indicoted includes both faces of the strip half-width a. Here we are concerned with the lower face, next to the dielectric. The actual shape in Fig. 2 or Fig. 6(a) has a certain effective half-width $g'-a'$ of the lower face as seen in Fig. 3(a). The special shape in Fig. 6(b) has a slightly greater effective half-width $\frac{1}{2}g''$ of the lower face as seen in Fig. 3(b). The excess s'' of the latter over the former is effective in the extreme of hi-k, so it is the parallel component of the shaded area in Fig. 3(a), as described with reference to Fig. 5.

$$s'' = \tfrac{1}{2}g'' - (g' - a') = (\tfrac{1}{2}g'' - a) - (g' - a' - a) \tag{22}$$

From (A-13), (A-14), (A-54), (A-57) [1], and still ignoring small differences (such as $d-g'$),

$$\begin{aligned} a &= (d-1) - \exp\text{-}a' - a' \\ &= g' - a' - 1 - \exp\text{-}a' \\ &= g' - a' - 1 - 1/2(d-1) \end{aligned} \tag{23}$$

$$g' - a' - a = 1 + \exp\text{-}a' = 1 + 1/2(d-1) \tag{24}$$

$$\begin{aligned} s'' &= \ln 4 - 1 - \exp\text{-}a' \\ &= 0.386 - \exp\text{-}a' \\ &= 0.386 - 1/2(d-1) \end{aligned} \tag{25}$$

In the limit of "very wide" strips,

$$\max s'' = \ln 4 - 1 = 0.386;$$

$$\frac{\max s''}{\max s'} = \frac{0.386}{0.903} = 0.427 \tag{26}$$

In this extreme, the parallel component is less than one-half the total shaded area in Figs. 3 and 5.

Formula (25) may be used to compute the parallel component s''/a' of the area fraction in the "wide" range. It is found to have a maximum value near the "square" shape, as shown in Fig. 7.

$$\max s''/a' = 0.125;$$

$$\frac{\max s''/a'}{\max s'/a'} = \frac{0.125}{0.201} = 0.622 \tag{27}$$

In this range, the parallel component is greater than one-half the total.

We now have formulas for the area fraction s'/a', resolved into its parallel and series components s''/a' and $(s'-s'')/a'$, for the range of "wide" strips. This includes the marginal range near the "square" shape. These formulas are graphed in Fig. 7, which will be explained further after the derivation for narrow strips. This concludes the conformal-mapping operations for wide strips, so we are ready for the application to mixed dielectric. The objective is the evaluation of the effective dielectric constant k' which is a mean between free-space (1) and the dielectric material k.

With reference to Figs. 3 and 5, the previous discussion has led to formulas for k' as weighted average between k and 1, in terms of the filling fraction. For wide strips, the derived values can be inserted in the general formulas such as (4) and (6) for computing k'; all the quantities needed in these formulas are evaluated in [1] or herein.

An example is given to indicate a procedure for computation. It is based on the "square" shape and a certain dielectric constant of the material. (See [1], Table I.)

$$a/b = 1; \qquad a = \pi$$

$$g' = d = 6.65$$

$$R_1 = 178 \text{ ohms}$$

$$a' = 2.416$$

$$a'/g' = 0.363$$

(16) $$s' = 0.482$$

(25) $$s'' = 0.297$$

. . . .

$$k = 2$$

(1) $$s = 0.390 = \tfrac{1}{2}s' + \tfrac{1}{2}s''$$

$$a' - s = 2.026 = 0.305\, g'$$

(4) $$q = 0.695$$

(6) $$k' = 1.695 \text{ (between 1.5 and 2)}$$

(9) $$R = 137 \text{ ohms}$$

This case is the center point on the square chart in Fig. 4.

Here we may compare the simple computation based on the lower bound of the filling fraction.

$$\tfrac{1}{2}g'' = \pi + \ln 4 = 4.528 \tag{21}$$

$$q = q'' = 0.681 \tag{7}$$

$$k' = 1.681 \tag{6}$$

This differs from the complete computation by only 0.01 of k' or 0.005 of R.

The present derivation enables a close approximation of the effective dielectric constant, over the range of "wide" strips, including the marginal range near the "square" shape. It is based on the conformal-mapping approximation of the preceding paper [1], from which the effective dielectric constant is here evaluated for the same shape. There remain to be given some explicit formulas for synthesis.

A formula for synthesis must be directed toward one of the properties which involve the dielectric, such as capacitance ($C \propto k'$), wave resistance ($R \propto 1/\sqrt{k'}$) or wave velocity ($\propto 1\sqrt{k'}$). The inductance L is independent of the dielectric. The wave resistance is selected as the most useful objective for synthesis; the shape ratio is to be computed after specifying the dielectric material k.

Therefore an approximate formula has been derived, which gives the shape ratio in terms of the wave resistance R and the dielectric constant k of the sheet material. From (8) and (9), the wave resistance is

$$R = \frac{R_c}{\sqrt{k'}}\frac{\pi}{g'} = \frac{R_c}{\sqrt{k'}}\frac{\pi}{d}; \qquad d = \frac{\pi}{\sqrt{k'}}\frac{R_c}{R} \tag{28}$$

We continue to ignore the small difference between g' and d. Here we define another parameter d_k which is the lesser value of effective half-width that would be required if the space were entirely filled with the dielectric material k.

$$R = \frac{R_c}{\sqrt{k}}\frac{\pi}{d_k}; \qquad d_k = \frac{\pi}{\sqrt{k}}\frac{R_c}{R} \tag{29}$$

The formula is to be a close approximation for "wide" strips (in this case $d_k > 2\pi$).

The required shape is between the limits that would be required for free space and for complete dielectric filling. For "wide" strips, it is much closer to the latter, which therefore is taken as a starting point. The derivation has been based on "very wide" strips, giving the exact terms for second approximation for both extremes of dielectric constant. The resulting approximate formula for synthesis is given in a form convenient for computation.

$$\frac{a}{b} = \frac{1}{\pi}(d_k - 1) - \frac{1}{\pi}\ln(2d_k - 1) + \frac{k-1}{2\pi k}\left[\ln(d_k - 1) + 0.293 - 0.517/k\right] \tag{30}$$

If $d_k > 2\pi$, estimated relative error < 0.01 of R.

In the limit of "very wide" strips, this formula is exact in the first two terms for the lo-k and hi-k extremes. It is an interpolation formula for intermediate conditions.

To obtain an explicit formula for analysis, (A-71) may be modified for mixed dielectric. The result is a second approximation for wide strips.

$$\begin{aligned}\frac{R}{R_c} &= \frac{\sqrt{1/k}}{\dfrac{a}{b} + \dfrac{1}{\pi}\ln 4 + \dfrac{k+1}{2\pi k}\ln\dfrac{\pi\epsilon}{2}\left(\dfrac{a}{b} + 0.94\right) + \dfrac{k-1}{2\pi k^2}\ln\dfrac{\epsilon\pi^2}{16}} \\ &= \frac{\sqrt{1/k}}{\dfrac{a}{b} + 0.441 + \dfrac{k+1}{2\pi k}\left[\ln\left(\dfrac{a}{b} + 0.94\right) + 1.451\right] + \dfrac{k-1}{k^2}(0.082)}\end{aligned} \tag{31}$$

In the denominator, the last term introduces the interpolation between the lower and upper bounds of the effective filling fraction; it has most effect for intermediate values of dielectric constant. It is estimated that the relative error is less than 0.01 for all conditions of "wide" strips.

V. Derivation for Narrow Strips

This derivation for narrow strips is based on free-space formulas and the principles of the foregoing derivation for wide strips. The free-space formulas for narrow strips are found in the literature, to the first approximation. The writer's unpublished derivation to the second approximation, including some properties of the free-space field, will be stated as required, without proof.

As an extension of the parameters already defined, some other are introduced which are better suited for narrow strips. These may be regarded as "separation" parameters h rather than "width" parameters g.

$$\begin{aligned} R_1 &= R_c h'/\pi = 120\, h'; \\ h' &= \pi R_1/R_c = R_1/120 = \pi^2/g' \end{aligned} \tag{32}$$

There are some related parameters that are defined here for reference. The average value between unity and the dielectric constant k of the sheet material is

$$k_a = \frac{k+1}{2} \tag{33}$$

If the space were filled with k_a, the wave resistance R would be given by the separation parameter

$$h_a = \pi\sqrt{k_a}\, R/R_c = \sqrt{k_a}\, R/120 \tag{34}$$

This will be used as a reference in some formulas.

Referring back to Fig. 1, the free-space wave resistance R_1 is given by the following formula to a second approximation for narrow strips:

$$\begin{aligned} h' &= \pi R_1/R_c \\ &= \ln\frac{4b}{a} + \frac{1}{8}\left(\frac{a}{b}\right)^2 \pm (\cdots)\left(\frac{a}{b}\right)^4 \pm \cdots \end{aligned} \tag{35}$$

The first approximation, for "very narrow" strips, may be derived by substituting for each strip a round wire which is equivalent in the first-order far field. The second approximation, for "narrow" strips, may be derived by substituting for each strip a spaced pair of round wires which is equivalent in the second-order far field. The later derivation, which is found to be rather simple, has been developed by the writer and verified by several alternative approaches.

On the same basis, the free-space formula for synthesis, to the second approximation, is also simple in explicit form.

$$\begin{aligned} b/a &= \tfrac{1}{4}\exp h' - \tfrac{1}{2}\exp\text{-}h' \\ &\quad \pm (\cdots)\exp\text{-}3h' \pm \cdots \end{aligned} \tag{36}$$

If $b/a > 1$, $h' > 1.485$ and relative error < 0.005 of R.

The marginal condition for narrow strips, as for wide strips, corresponds roughly to $R = \frac{1}{2}R_c/\sqrt{k'}$, for which $h' = \pi/2$.

Referring to Fig. 2, the fields around "very narrow" strips can be computed to a first approximation by replacing each strip by an equivalent small round wire. Several simple relations become apparent, as will be stated.

In Fig. 2, there is a dashed curve between points ③ and ⑦ which represents the free-space flux line terminating at the edge of the strip. This flux line approaches the shape of a circular arc (quadrant). As for the effect of the dielectric, we note that most of the electric energy is close to the strip, and is divided nearly equally between both faces of the strip. Therefore the effective dielectric constant, for "very narrow" strips, approaches the lower limit,

$$k' = k_a = \frac{k+1}{2} \tag{37}$$

The excess over this value is caused by the shaded area, whose effect increases with strip width.

Referring to Fig. 3(a), the flux fraction on the outer faces may be based on the free-space field. It is found to have the following value, a second approximation for "narrow" strips.

$$\begin{aligned} \frac{a'}{g'} &= \frac{1}{2} - \frac{1}{2\pi}\operatorname{antitan}\frac{a}{b} \\ &= \frac{1}{2} - \frac{2}{\pi}\exp\text{-}h' + \frac{20}{3\pi}\exp\text{-}3h' \pm \cdots \end{aligned} \tag{38}$$

In this series, the first two terms are exact. The deficiency from $\frac{1}{2}$ decreases rapidly with decreasing width or with increasing separation and wave resistance. While this flux fraction is of interest for narrow strips, it will not be needed for the first approximation to be developed here.

In Fig. 3(a), the dielectric boundary is represented by the curve between points ③ and ⑧. From the small-strip or small-wire field in free space, this curve is found to be

$$y' = \pi - \frac{g'}{\pi}\ln\frac{1}{\cos\dfrac{\pi x'}{g'}} \tag{39}$$

In the entire rectangle ($0 < x' < g'$; $0 < y' < \pi$), the area fraction above the curve is the filling fraction, and may be formulated as follows:

$$q' = \frac{1}{2} + \frac{\ln 2}{2\pi^2}g' = \frac{1}{2} + \frac{\ln 2}{2h'} \tag{40}$$

The "parallel" component of this filling fraction is to be evaluated with reference to Figs. 3, 5, and 6, as was done for wide strips. The properties of a narrow strip between two planes are given in the Appendix. The width parameter may be formulated from (56) and (57), as follows, to a first approximation for very narrow strips.

$$\begin{aligned} g'' &= \frac{\pi^2}{\ln\dfrac{4b}{a} - \ln\dfrac{\pi}{2}} = \frac{g'}{1 - \dfrac{\ln \pi/2}{\pi^2}g'} \\ &= g' + \frac{\ln\pi/2}{\pi^2}g'^2 \end{aligned} \tag{41}$$

From this, the parallel part of the filling fraction may be evaluated.

$$q'' = \frac{1}{2}g''/g' = \frac{1}{2} + \frac{\ln \pi/2}{2h'} \tag{42}$$

As before, the excess of the entire filling fraction over the parallel component is taken to be the series component, and may be evaluated as follows:

$$q' - q'' = \frac{\ln 4/\pi}{2h'} \tag{43}$$

From these formulas, by interpolation as in (4), the effective filling fraction is

$$q = q'' + \frac{1}{k}(q' - q'')$$
$$= \frac{1}{2} + \frac{1}{2h'}\left(\ln \pi/2 + \frac{1}{k}\ln 4/\pi\right) \tag{44}$$

Then the effective dielectric constant is

$$k' = \frac{k+1}{2} + \frac{k-1}{2h'}\left(\ln \pi/2 + \frac{1}{k}\ln 4/\pi\right)$$
$$= k_a\left[1 + \frac{k-1}{k+1}\,\frac{\ln \pi/2 + \frac{1}{k}\ln 4/\pi}{\ln 4b/a + \cdots}\right] \tag{45}$$

Over the range of narrow strips ($b/a > 1$) the second term is greatest for hi-k, but is still less than $\frac{1}{4}$ of the total.

In the limit of very narrow strips (so $a' = \frac{1}{2}g'$) these relations may be expressed in terms of effective width as was done for wide strips.

$$s' = \frac{2\ln 2}{\pi^2}a'^2; \qquad s'' = \frac{2\ln \pi/2}{\pi^2}a'^2 \tag{46}$$

This form will be used for computing the slope at the "narrow" end of the curves in Fig. 7.

Here again, an approximate formula for synthesis has been derived to give the shape ratio for specified wave resistance R and dielectric constant k. The required separation parameter h' is formulated in terms of the hypothetical parameter h_a for k_a.

$$h' = h_a + \frac{1}{2}\,\frac{k-1}{k+1}\left(\ln\frac{\pi}{2} + \frac{1}{k}\ln\frac{4}{\pi}\right)$$
$$= h_a + \frac{\ln 2 + (k-1)^2 \ln \pi/2}{2k(k+1)}$$
$$= h_a + \frac{k-1}{k+1}(0.226 + 0.120/k) \tag{47}$$

Then the shape ratio can be computed by (36). If $b/a > 1$, the second term is less than $\frac{1}{10}$ as great as the first, and the estimated relative error is within 0.005 of h' or R.

As in the case of wide strips, we can obtain an implicit formula for analysis, which is a second approximation for narrow strips.

$$\frac{R}{R_c} = \frac{1}{\sqrt{k_a}}\,\frac{1}{\pi}\left[\ln\frac{4b}{a} + \frac{1}{8}\left(\frac{a}{b}\right)^2 - \frac{1}{2}\,\frac{k-1}{k+1}\left(\ln\frac{\pi}{2} + \frac{1}{k}\ln\frac{4}{\pi}\right)\right] \tag{48}$$

It is estimated that the relative error is less than 0.01 for all conditions of narrow strips.

VI. Relations for Wide and Narrow Strips

Having derived various formulas for wide strips or narrow strips, it is interesting and instructive to note the behavior over the entire range of the shape ratio a/b or b/a. The end objective is to present useful design information in convenient form. Figure 9 is intended to serve this purpose; it is introduced here for reference in the discussion within this section. Figure 9(a), (b), and (c) give in three ranges a graphical presentation of the relation between wave resistance and shape ratio, with dielectric constant as a parameter. Figure 9(d) gives the relation between the shape ratio and the effective filling fraction, which will be discussed further on in this section. [The four parts of this figure are arranged in clockwise order for showing (a), (b), and (c) in logical sequence, preserving the proximity of adjoining regions.]

Figure 4 has been introduced as a map of the entire range of parameters. Each scale is chosen to locate the transition condition in the vicinity of the center. The scale of ordinates is

$$\frac{k-1}{k} = 1 - 1/k \tag{49}$$

A scale of abscissas is chosen to locate the "square" shape on the centerline.

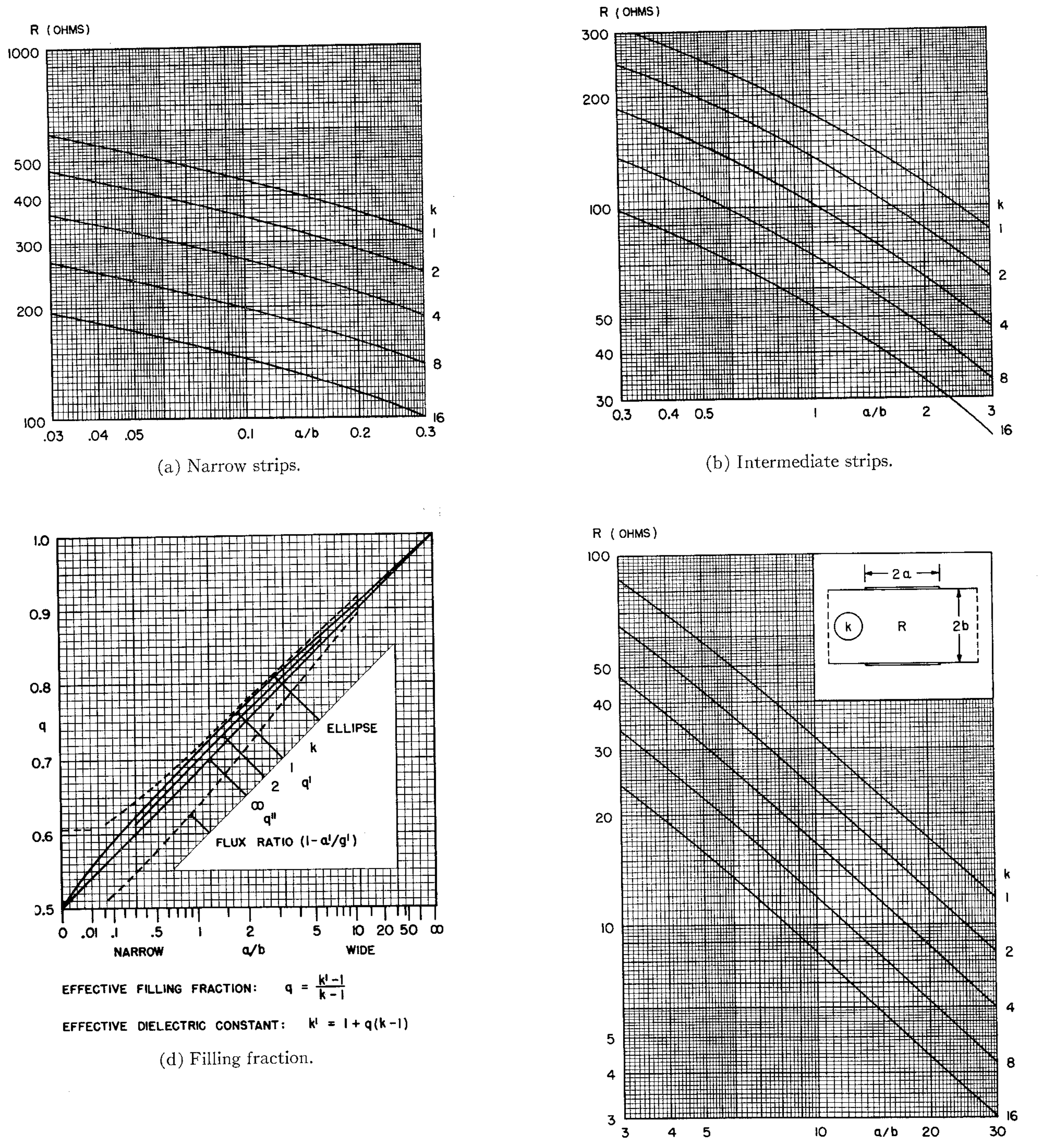
The effective filling fraction q is a significant parameter for distinguishing the various limits or bounds over the map in Fig. 4. It has a particular designation and definition on each of the four sides of the square, and it determines the effective dielectric constant k'.

On the lower border, the effective filling fraction is equal to the actual filling fraction q' based on Figs. 3(a) and 5. On the upper border, the effective filling fraction is equal to the parallel part q'' based on Figs. 3 and 5. These provide upper and lower bounds which are rather close together. Each of these is susceptible of exact description and is susceptible of approximate or exact formulation. These are related to the bounds of area fraction graphed in Fig. 7.

Contours of effective dielectric constant k' may be mapped over the region shown in Fig. 4. One contour is shown, the one through the center.

The center point is particularly interesting, as a midway interpolation between the extremes. At the center, where $k = 2$, it is found that $k' = 1.70$. The corresponding values of k at the ends of this contour (2.4 and 1.7) have substantially equal displacement above and below the horizontal centerline. Therefore, this contour can be drawn as a straight line, which then determines a particular scale of abscissas.

In this family of contours k', the bottom and top contours are horizontal straight lines. Therefore, it is likely that all the contours would be nearly straight lines having slopes between zero and that of the contour through the center.



(a) Narrow strips.

(b) Intermediate strips.

(d) Filling fraction.

(c) Wide strips.

Fig. 9. Graphical charts for design.

The limits of the effective filling fraction are here determined by evaluation and interpretation of the shaded area in Figs. 3 and 5. This has been formulated in terms of the area fraction graphed in Fig. 7. Referring to the dielectric-boundary curve and to the rectangle in which it is inscribed, the area fraction s'/a' is the ratio of the shaded area (outside of the curve) over the area of the rectangle, as previously defined. This fraction has been divided into two components, the parallel component s''/a' and the series component $s'/a-s''/a'$. Furthermore, the interpolation formula (1) gives the effective area fraction s/a' which depends on the dielectric constant k. These are graphed on a scale of abscissas t which has been chosen for reasons to be developed here.

In general terms, it has been found that the area fraction s'/a' approaches zero at both extremes of shape. Furthermore, it is found that the area fraction is simply related to the parameter a', by direct and inverse proportionalities for the narrow and wide extremes.

Referring to Fig. 7, the following table identifies the equations for evaluating the end slopes and the maximum value of each of the extreme curves.

Shape	Graph	Area Fraction	
		s'/a'	s''/a'
Very narrow strips	Slope	(46)	(46)
Intermediate strips	Maximum	(20)	(27)
Very wide strips	Slope	(15)	(26)

It is desired to graph these area fractions on a scale that will give the curves a simple shape. The following variable for the scale of abscissas gives each curve a shape similar to a sine wave:

$$t = \frac{2}{\pi} \text{ antitan } a'/a^* \tag{50}$$

The center point is located at the "square" shape by choosing $a^* = 2.416$ [1], (Table I). The peak of each curve is then near this centerline.

The effective area fraction s/a' is obtained by interpolation in accordance with (1), depending on k. It is graphed for several values of k giving equal steps of $1/k$.

In Fig. 7, it is seen that the area fraction has a maximum value slightly less than that corresponding to an elliptic quadrant inscribed in the same rectangle. Toward the extremes, the boundary curve approaches the rectangle so the area fraction approaches zero. The scale of the shape ratio a/b is seen to be crowded towards the center so that, over the usual range of shape, the area fraction is near the maximum. Therefore the maximum values, which are noted on the right-hand side, may be taken as a useful approximation.

The effective filling fraction q is particularly significant in that it is determined mainly by the shape ratio and is only slightly dependent on the dielectric constant. From its value, it is easy to compute the effective dielectric constant. For reference, the effective filling fraction is graphed in Fig. 9(d). The scale of shape ratio is determined to give one limit q'' as a straight line. This facilitates a comparison of the slightly different curves.

In Fig. 9(d), the principal (solid-line) graphs are those for the extremes of k and one intermediate value. The lowest (dashed-line) graph is a lower bound set by the free-space flux ratio. The uppermost (dashed-line) graph is an upper bound set by the ellipse, as seen in Fig. 7. The problems and solutions presented herein are best appreciated and understood by reference to this set of graphs.

It would be convenient if the effective filling fraction were dependent entirely on the shape ratio and independent of the dielectric constant, so a single relationship would suffice. The lowest graph of q is q'', which is effective in the limit of hi-k. This comes near to the ideal of a single graph independent of k. If q'' is taken for q, this causes a relative error less than 0.01 of k' or 0.005 of R. This is close enough for practical purposes. Also it happens that this case is based on the strip between two planes, which is the easiest to formulate. As previously mentioned, this concept has been presented by Dukes [3], [4] but it seems that he did not correctly apply it to the problem at hand.

The dashed curves in Fig. 9(d) show the brackets that could be most simply formulated by conformal mapping of the dielectric boundary. The interpolation between these brackets is the subject of Fig. 7.

While the present derivations are limited to thin strips, a small thickness can be compensated by a reduction of width. While retaining the same separation $2b$, each strip is taken to have a rectangular cross section of small thickness Δb less than the half-width a and much less than the half-separation b. Each edge recedes by a small amount Δa, which is the edge correction to retain the same properties.

This edge correction is simply formulated for free-space ($k=1$) as follows, for wide and narrow strips:

$$\text{If } a > b/4\pi > \Delta b: \quad \Delta a = \frac{\Delta b}{2\pi}\left(\ln\frac{2b}{\Delta b} + 1\right) \tag{51}$$

$$\text{If } b/4\pi > a > \Delta b: \quad \Delta a = \frac{\Delta b}{2\pi}\left(\ln\frac{8\pi a}{\Delta b} + 1\right) \tag{52}$$

In the range of validity, $\Delta a > \frac{2}{3}\Delta b$.

With the dielectric sheet $k>1$, the edge recession has more effect because it decreases the area in contact with dielectric while the thickness merely increases the area exposed to empty space. In the extreme of hi-k, the edge correction approaches zero. It is proposed that the edge correction Δa in these formulas be multiplied by $1/k$, as a sensible interpolation between the extremes.

VII. Procedures for Computation

The shape ratio determines the free-space wave resistance R_1 and is the principal factor in determining the effective filling fraction q. The latter and the dielectric constant k determine the effective dielectric constant k'. Then R_1 and k' determine the actual wave resistance R. These principles underlie any computation in the present configuration including mixed dielectric. For each case, a procedure is sought which will include a sequence of explicit formulas.

The effective filling fraction q'' for hi-k is particularly significant and is simply defined by (7). For practical purposes, this parameter determines the effect of the dielectric sheet. It may be included in various sequences of computation.

Closer computation for theoretical purposes requires also the actual filling fraction q' which is effective for lo-k. This parameter may be computed from various approximations given for wide or narrow strips. Then (4) gives the effective filling fraction q for any k.

From the derivation for either wide or narrow strips, a complete computation procedure can be gleaned for the closest approximation. There may not be a sequence of explicit formulas, starting with any two of the three essential parameters R, a/b, k, in which case there is the alternative of starting with an intermediate parameter such as R_1. The latter may be required for wide strips, as in [1].

The transition between the conditions of wide and narrow strips, for the purpose of selecting the closer formulas, is usually on the narrow side of the square condition. As a rule, this transition may be associated with the shape, $a/b=\frac{1}{2}$. In particular, this is near the cross-over of the approximate formulas for a strip between two planes.

There have been presented some explicit formulas for R of a/b, or the reverse, including k as a parameter. These are adequate for practical purposes, so they are listed here.

Range of Shape	Synthesis	Analysis
Wide strips	(29)(30)	(31)
Narrow strips	(34)(47)(36)	(48)

VIII. Design Charts

Figure 9 is a graphical presentation of the results of this study, in a form adequate for practical purposes.

Figure 9(a), (b), and (c) gives, in three sections, the relation between R and a/b for k in binary steps from 1 to 16. Figure 9(a) includes the range covered by the complete formulas for narrow strips; Fig. 9(b) and (c), the range covered by the complete formulas for wide strips.

Figure 9(d) gives the relation between the effective filling fraction q and the shape ratio a/b, including the small dependence on k. From this ratio, (6) enables simple computation of k'.

The velocity ratio for any shape is equal to $1/\sqrt{k'}$. It may be taken off the R curves, since it is equal to the resistance ratio R/R_1 for any shape.

As previously mentioned, the form of the R curves is patterned after Dukes [4], but his curves have discrepancies up to 0.15 of R. The present curves cover a wider range of shape, on a more expanded scale.

IX. Conclusion

In this problem of mixed dielectric, it has been found possible to compute the effective dielectric constant to a remarkably close approximation. It is placed between rather close limits that are exactly defined and approximately formulated. Then it is evaluated by interpolation between these limits, using a rule that has a logical basis in principle and seems sensible for this application.

This approach has been applied to the case of parallel-strip conductors on the faces of a dielectric sheet. The result is a formulation of this case over the entire range of the shape ratio and the dielectric constant of the sheet material. For synthesis or analysis, there are given some simple explicit formulas capable of approximation within about one per cent of wave resistance.

X. Symbols

k = dielectric constant of sheet of material separating the pair of strips.

k' = effective dielectric constant of all space around the pair of strips.

$k_a=(k+1)/2$ = average of dielectric constants of sheet and free space.

q' = entire filling fraction of dielectric material.

q'' = parallel part (nearly all) of filling fraction.

$q=(k'-1)/(k-1)$ = effective filling fraction.

$R_c=377$ ohms = wave resistance of square area of free space.

R = wave resistance of symmetrical pair of strips on dielectric sheet k; or of one quadrant of its cross section.

R_1 = wave resistance of pair of strips in free space ($k=1$).

C = capacitance.

L = inductance.

ϵ_0 = electrivity (electric permittivity) of free space.

μ_0 = magnetivity (magnetic permeability) of free space.

l = length of pair of strips (for C or L).

$x+jy=z$ = complex plane of space coordinates, the cross section of the pair of strips.

$x'+jy'=z'$ = complex plane of flux-potential coordinates.

a = half width of strip conductor.

b = half separation of parallel strips.

a/b = shape ratio.

a' = effective half width of outer face of strip.

g' = effective half width of strip, including flux on both outer and inner faces.

g'/π = ratio of effective width over separation.

a'/g' = fraction of total flux that terminates on outer face of strip.

g'' = effective half width of strip between parallel planes.

$\Delta a=g''-a$ = effective increment of half width of "wide" strip between parallel planes.

$a^*=2.416=a'$ for "square" shape ($a/b=1$).

d = parameter defined in [1], differing slightly from g' for "wide" strips.

d_k = parameter d for same R but all space filled with dielectric k.

$h'=\pi^2/g'$ = separation parameter.

$h'/\pi=\pi/g'$ = ratio of half separation over effective half width.

h_a = parameter h' for same R but all space filled with average dielectric k_a.

s' = increment of effective half width, including parallel and series parts.

s'' = parallel part of s'.

s = interpolated increment of effective half width for any dielectric constant k.

t = special scale of abscissas, related to shape ratio.

Appendix

A Strip Between Parallel Planes

An essential part of this presentation is the relation between the subject shape and a slightly different shape, compared with reference to (a) and (b) in Figs. 3 and 6. The different shape is a thin strip midway between two parallel planes, for comparison with a strip spaced from one plane. The properties of this different shape are summarized here for reference where needed in the text.

Figure 6(a) shows the upper two quadrants of the subject shape, including one of the two strips and the neutral plane below. Figure 6(b) shows the same region around one strip, but with the addition of a parallel second plane equally spaced above the strip. The free-space properties of the latter are to be formulated here.

The derivation for this shape is well known, in terms of complete elliptic integrals. These can be closely approximated for the cases of wide or narrow strips by simple formulas in terms of slide-rule functions.

Referring to Fig. 6(b), the effective width of the strip is greater than twice the actual width because both sides are exposed to nearby planes. In one quadrant, the effective width g'' is greater than twice the half-width a by twice the increment of width Δa. The well-known derivation, applied to the one quadrant, gives

$$g''/b = g''/\pi = \pi/h'' = R_c/R'' = 2K/K' \tag{53}$$

in which

$$K = K(\sin\alpha); \qquad K' = K(\cos\alpha) \tag{54}$$

$$\sin\alpha = \tanh a/2; \quad \cos\alpha = \operatorname{sech} a/2 \tag{55}$$

$$\tan\tfrac{1}{2}\alpha = \tanh\tfrac{1}{4}a = \tanh\frac{\pi a}{4b} \tag{56}$$

Here, the free-space wave resistance in one quadrant is R''. The complete elliptic integral K is defined and tabulated in Dwight.

The ratio K/K' can be closely approximated over each of two ranges, corresponding to narrow and wide strips. Applying these approximations, we obtain the following simple formulas.

Narrow strips; $\alpha < \tfrac{1}{4}\pi$; $a/b < 0.56$;

$K/K' < 1$; $R'' > \tfrac{1}{2}R_c$:

$$g'' = \frac{\pi^2}{\ln\dfrac{2}{\tan\frac{1}{2}\alpha}} = \frac{\pi^2}{\ln\dfrac{2}{\tanh\frac{1}{4}a}} < 2\pi \tag{57}$$

Wide strips; $\alpha > \tfrac{1}{4}\pi$; $a/b > 0.56$;

$K/K' > 1$; $R'' < \tfrac{1}{2}R_c$:

$$g'' = 4\ln\frac{2}{\tan(\frac{1}{4}\pi - \frac{1}{2}\alpha)} = 2(a + \ln 4) > 2\pi \tag{58}$$

Relative error <0.003 of g'' (or 0.005 of a).

The approximation for wide strips is in a well-known form, including $\Delta a = \ln 4$, as given by Maxwell [2]. The corresponding approximation for narrow strips has the same mathematical basis, but is not so well known.

The transition between the two cases is not critical; it may be made at $a/b = 0.5$, which is near $g'' = 2\pi$. In terms of a strip and one plane, this is near $g' = \tfrac{3}{2}\pi$.

While not essential to this presentation, there are given here for reference the relations between wave resistance and shape between two planes.

Narrow strips (as before):

$$R_1 = R_c\frac{1}{2\pi}\left[\ln\frac{8b}{\pi a} + \frac{\pi^2}{48}\left(\frac{a}{b}\right)^2 - \cdots\right] \tag{59}$$

$$\frac{b}{a} = \frac{\pi}{8}\exp 2\pi R_1/R_c - \frac{\pi}{6}\exp - 2\pi R_1/R_c + \cdots \tag{60}$$

Wide strips (as before):

$$R_1 = R_c\frac{\frac{1}{2}}{\dfrac{a}{b} + \dfrac{1}{\pi}\ln 4} + \cdots \tag{61}$$

$$\frac{a}{b} = \frac{1}{2}R_c/R_1 - \frac{1}{\pi}\ln 4 + \cdots \tag{62}$$

Each of these formulas includes the first and second terms of a rapidly convergent series. They are given in explicit form for analysis and for synthesis.

Acknowledgment

This study and presentation have been stimulated by the needs of a printed-circuit array antenna being designed for Hazeltine Corporation. The writer is grateful to Mrs. Nancy M. Staron for the careful typing of the extensive mathematical content of this manuscript, and to Henry J. Newton for the skillful preparation of the graphs.

References

[1] Wheeler, H. A., Transmission-line properties of parallel wide strips by a conformal-mapping approximation, *IEEE Trans on Microwave Theory and Techniques*, vol MTT-12, May 1964, pp 280–289.

[2] Maxwell, J. C., *Electricity and Magnetism*, 3 ed. London: Oxford University Press, 1892. (Edge correction for a wide strip between parallel planes, p 303.)

[3] Dukes, J. M. C., Properties of strip transmission lines with the aid of an electrolytic tank, *Proc. IEE*, vol 103, pt. B, May 1956, pp 319–333. (Rough computation for mixed dielectric in case of strip and plane separated by sheet.)

[4] ——, The application of printed-circuit techniques to the design of microwave components, *Proc. IEE*, vol 105, pt. B, Aug 1958, pp 155–172. (Graphs of wave resistance and velocity ratio for parallel strips separated by dielectric sheet, subject to errors.)

[5] Frazita, R. F., Transmission line properties of coplanar parallel strips on a dielectric sheet, M.S. thesis, Polytech. Inst. of Brooklyn, N. Y., 1965.